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# Domain walls in non-Abelian gauge theory

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**Abstract.** In a five-dimensional version of non-Abelian gauge theory the existence is demonstrated of *domain walls* trapping particle excitations in the extra dimension, compactified of $S^1$. Domain walls are soliton-like solutions of the Nahm-Bogomolny equations for a Yang-Mills system dimensionally reduced to $(1+1)$ dimensions. The soliton-like solutions can also be interpreted either as *strings* in a $(2+1)$-dimensional Yang-Mills-Higgs model or as *membranes* in a $(3+1)$-dimensional pure Yang-Mills theory.

## 1. Introduction

In a very interesting paper [1], Rubakov and Shaposhnikov have proposed an unusual procedure for dimensional reduction. Instead of compactifying extra dimensions as in standard schemes based on the original Kaluza-Klein idea [2], they deal with a flat $(3+N+1)$-dimensional spacetime; particles are confined in the extra $N$ dimensions by a potential well arising from a domain wall but are free in Minkowski space, $\mathbb{R}^{3,1}$. The $\lambda\Phi^4$ model in $(3+1+1)$ dimensions illustrates such a situation, where the potential well coming from the domain walls corresponds to the kink solution in the extra dimension.

In this paper we consider a $(3+1+1)$-dimensional Yang-Mills system sharing characteristics of both the Rubakov-Shaposnikov and Kaluza-Klein procedures. The model presents 'periodic' kinks as solutions of the Bogomolny-Nahm [3] equations in the extra dimension, which in our case is angular, producing domain walls in the real world. They appear as periodic trajectories in the fundamental subsystem of Saviddy's Yang-Mills classical mechanics [4] and at the limit where the radius of the ring in the extra dimension tends to infinity, the situation covered by Rubakov and Shaposhnikov in the $\lambda\Phi^4$ model, they form the separatrix from unbounded motion.

In the WKB approximation the particle excitations are given by the spectrum of the Hessian at the periodic kink solutions; the Hessian in the extra coordinate proves to be related to the differential operator of the Lamé equation for $n=1$ [5]. Variations in the remaining coordinates lead to free propagation in Minkowski space and, restricting ourselves to real eigenvalues of the Lamé equation, three kinds of excitations exist. (a) Particles trapped by potential wells $\mathcal{V}_\alpha = 2k^2\,\mathrm{sn}^2\,\varphi - d_\alpha$, which are free particles in Minkowski space. (b) Particles with higher energy than $\mathcal{V}_\alpha$ which escape through the $\varphi$ dimension. (c) Tachyons forced by the topology of the model via the Ljusternik-Schnirelman theory [6].

Another completely different physical interpretation for this kind of soliton-like solution can be given; the solutions can be thought of either as strings in the $(\mathrm{YMH})_{2+1}$

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model or as membranes of finite thickness in the pure $(\mathrm{YM})_{3+1}$ model. Via the Wilson criterion they provide a mechanism for confinement of both colour-electric and colour-magnetic charges in $(2+1)$-dimensional gauge theories. The second alternative shows the existence of membranes in non-Abelian gauge theories from which one can obtain an extremely rich particle spectrum by considering its quantum fluctuations. The dimensional reduction process is slightly different from that previously described; it is convenient to put space and time dimensions on the same footing; to consider periodic boundary conditions and only at the end to choose the time coordinate by analytic continuation to the imaginary axis.

Fermions will be analysed by solving the Dirac equation in the background of the domain walls. Fermionic zero modes will exist, which freeze the fermion propagation in the extra dimension or cause a strong violation of the fermion number by a mechanism reminiscent of the Witten proposal for superconducting strings [7].

The paper is organised as follows. In section 2 we describe the dimensional reduction to be applied to the gauge field contribution in the QCD action on a five-dimensional (infinite) cylinder. At the same time, a similar dimensional reduction for the $\lambda\Phi^4$ model, just for the sake of comparison, is also explained. In section 3 periodic kink solutions are found by solving first-order equations arising as Bogomolny equations for the reduced system. They are expressed in terms of elliptic functions but an analysis of their hyperbolic limit and its physical origin is performed. The dependence on a steepness parameter is also explicitly shown. Section 4 is devoted to studying the particle spectrum. We also unveil the topological origin of the tachyonic excitations and suggest a loophole for avoiding them. In section 5 we discuss the different appearances as physical objects of the periodic kink, i.e. membranes or strings, depending on the dimensional reduction scheme chosen. Finally, in section 6 the effects of our solutions on fermions are briefly considered. We conclude with some remarks about the possibility of similar solutions in other physical models.

## 2. Dimensional reduction and Bogomolny equations

We shall start by showing how our mechanism of dimensional reduction works in the $(3+1+1)$-dimensional $\lambda\Phi^4$ model given by the action

$$S=\int \mathrm{d}\underset{\sim}{x}^5[\tfrac{1}{2}\underset{\sim}{\partial}_M\underset{\sim}{\Phi}\underset{\sim}{\partial}^M\underset{\sim}{\Phi}-\tfrac{1}{4}\lambda(\underset{\sim}{\Phi}^2-m^2/\lambda)^2] \tag{2.1}$$

$$M=\{0,1,2,3,4\} \qquad g_{00}=-g_{ii}=-g_{44}=+1 \qquad g_{MN}=0 \quad \text{if} \quad M\neq N.$$

We introduce dimensionless variables, $\underset{\sim}{\Phi}=(m/\sqrt{\lambda})\Phi$ and $\underset{\sim}{x}_M=(1/m)x_m$. The action becomes

$$S=\frac{1}{m\lambda}\int \mathrm{d}^5x[\tfrac{1}{2}\partial_M\Phi\partial^M\Phi-\tfrac{1}{4}(\Phi^2-1)^2]$$

and in the case where the scalar field $\Phi$ depends only on the $x_4$ coordinate, it reduces to $S=L^3TE(m,\lambda)$, where $L^3$ and $T$ are respectively normalisation volume and time, and $E(m,\lambda)$ is

$$E(m,\lambda)=\frac{m^3}{\lambda}\int \mathrm{d}x_4\left[\frac{1}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x_4}\right)^2+\tfrac{1}{4}(\Phi^2-1)^2\right]. \tag{2.2}$$

The variational problem defined by $E$ has a first integral, the 'particle' energy

$$C=\frac{1}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x_4}\right)^2-\tfrac{1}{4}(\Phi^2-1)^2. \tag{2.3}$$

We shall 'renormalise' $C$ to zero by introducing an elliptic parameter $k^2$, related to the 'particle' energy by $C=\frac{1}{2}(1-k^2)^2/(1+k^2)$, and defining a new 'particle' coordinate $\phi=[2k^2/(1+k^2)]^{1/2}\Phi$ and a new 'particle' time $\varphi=(1/\sqrt{1+k^2})x_4$†. Then (2.3) reduces to

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}\right)^2-(1-\phi^2)(1-k^2\phi^2)=0 \tag{2.4a}$$

or

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}-(1-\phi^2)^{1/2}(1-k^2\phi^2)^{1/2}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}+(1-\phi^2)^{1/2}(1-k^2\phi^2)^{1/2}\right)=0. \tag{2.4b}$$

The second form, (2.4$b$), leads to a generalised version of the famous Bogomolny equation for solitons [8]. We could also obtain (2.4) in a closer way to the original by writing $E(\lambda, m)$ in the form

$$\begin{aligned}E(\lambda, m)=\frac{m^3}{\lambda(1+k^2)^{1/2}}\bigg[\int \mathrm{d}\varphi\,\frac{1}{2}\left(\frac{2k^2}{1+k^2}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}\mp(1-\phi^2)^{1/2}(1-k^2\phi)^{1/2}\right)^2\\ \pm\int \mathrm{d}\varphi\left(\frac{2k^2}{1+k^2}\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}(1-\phi^2)^{1/2}(1-k^2\phi^2)^{1/2}\mp\frac{1}{2}\frac{(1-k^2)^2}{1+k^2}\right)\bigg].\end{aligned} \tag{2.5}$$

The solutions of (2.4) are the absolute minima of $E(\lambda, m)$ for a given $k$. We now move to the $(3+1+1)$-dimensional QCD action

$$\begin{gathered}S=\frac{1}{2}\int \mathrm{d}^5\underset{\sim}{x}\,\mathrm{tr}\,\underset{\sim}{F}_{MN}\underset{\sim}{F}^{MN}+\int \mathrm{d}^5\underset{\sim}{x}\,\bar{\psi}(x)(\mathrm{i}\gamma_M\underset{\sim}{D}^M+m)\underset{\sim}{\psi}(x)\\ \underset{\sim}{F}_{MN}=\underset{\sim}{\partial}_M\underset{\sim}{A}_N-\underset{\sim}{\partial}_N\underset{\sim}{A}_M-\mathrm{i}g[\underset{\sim}{A}_M,\underset{\sim}{A}_N] \qquad \underset{\sim}{D}_M=\underset{\sim}{\partial}_M-\mathrm{i}g[\underset{\sim}{A}_M\\ \underset{\sim}{F}_{MN}=\underset{\sim}{F}^a_{MN}T^a \qquad (T^a)^+=-T^a \qquad a=1,2,\ldots,8 \qquad \gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3\end{gathered} \tag{2.6}$$

for the SU(3) gauge group. We also introduce dimensionless variables $\underset{\sim}{A}=(m/g)A$ and $\underset{\sim}{x}_M=(1/m)x_M$ and the pure gauge action becomes

$$S=\frac{1}{2mg^2}\int \mathrm{d}^5x\,\mathrm{tr}\,F_{MN}F^{MN}. \tag{2.7}$$

In the case where the gauge potential $A_M$ do not depend on the spatial coordinates $x_i$ the action $S$ can be written as $S=L^3I(g, m)$ where $I(g, m)$ is, in the temporal gauge $A_0=0$, given by

$$\begin{gathered}I(g, m)=\frac{m^2}{2g^2}\int \mathrm{d}x_0\int \mathrm{d}x_4(-\mathrm{tr}\,F_{04}F_{04}+\mathrm{tr}\,F_{i4}F_{i4}+\mathrm{tr}\,F_{ij}F_{ij})\\ F_{04}=\partial_0A_4 \qquad F_{i4}=-\partial_4A_i-\mathrm{i}[A_i, A_4] \qquad F_{ij}=-\mathrm{i}[A_i, A_j].\end{gathered} \tag{2.8}$$

† The 'particle' energy, coordinate and time are not to be confused with the field energy, $E(\lambda, m)$, field coordinates, $x_i$, and field time, $x_0$. The reason for using these names is the dimensional reduction we will perform below.

$I(g, m)$ is the action for a field theory in $(1+1)$ dimensions with field equations

$$\begin{aligned} &\partial_0\partial_4 A_4 = 0 \qquad \partial_0^2 A_4 + [A_i D_4 A_i] = 0 \\ &\partial_4^2 A_i - \mathrm{i}[\partial_4 A_4, A_i] - [A_4, [A_4, A_i]] - \partial_0^2 A_i = 0. \end{aligned} \tag{2.9}$$

Static solutions of (2.9) are critical points of the function $E(g, m)$

$$E(g, m) = \frac{m^3}{2g^2} \int \mathrm{d}x_4 (2 \operatorname{tr} D_4 A_i D_4 A_i - \operatorname{tr}[A_i, A_j][A_i, A_j]) \tag{2.10}$$

such that for solutions of

$$\frac{\delta E}{\delta A_4}(g, m) = [A_i, D_4 A_i] = 0 \qquad \frac{\delta E}{\delta A_i}(g, m) = \mathrm{D}_4^2 A_i = 0 \tag{2.11}$$

the action is $L^3 T E(g, m)$. Notice that $E(g, m)$ is the Saviddy 'action' for YM classical mechanics with 'Euclidean' time [4]. In the axial gauge $A_4 = 0$ the first integral of the 'particle' energy has the form

$$\varepsilon = \operatorname{tr} \frac{\mathrm{d}A_i}{\mathrm{d}x_4} \frac{\mathrm{d}A_i}{\mathrm{d}x_4} - \tfrac{1}{2} \operatorname{tr}[A_i, A_j][A_i, A_j] \tag{2.12a}$$

or

$$\varepsilon = \operatorname{tr}\left(\frac{\mathrm{d}A_i}{\mathrm{d}\varphi} - \varepsilon_{ijk}[A_j, A_k]\right)\left(\frac{\mathrm{d}A_i}{\mathrm{d}\varphi} + \varepsilon_{ijk}[A_j, A_k]\right) \tag{2.12b}$$

in terms of the time $\varphi$ and gauge potentials rescaled by $(1+k^2)^{-1/2}$. As in the scalar case, $E(g, m)$ may be written *à la* Bogomolny

$$\begin{aligned} E(g, m) = \frac{m^3}{2g^2}\bigg\{ &\int \mathrm{d}\varphi \left[ 2 \operatorname{tr}\left(\frac{\mathrm{d}A_i}{\mathrm{d}\varphi} \mp \varepsilon_{ijk}[A_j, A_k]\right)\left(\frac{\mathrm{d}A_i}{\mathrm{d}\varphi} \mp \varepsilon_{ijk}[A_j, A_k]\right)\right] \\ &\pm 4 \int \mathrm{d}\varphi \left(\varepsilon_{ijk} \frac{\mathrm{d}A_i}{\mathrm{d}\varphi}[A_j, A_k]\right)\bigg\} \end{aligned} \tag{2.13}$$

and the value $\varepsilon = 0$ is attained by solutions of

$$\frac{\mathrm{d}A_i}{\mathrm{d}\varphi} = \pm \varepsilon_{ijk}[A_j, A_k]. \tag{2.14}$$

The set of equations (2.14) are thus the Bogomolny equations for the $(1+1)$-dimensional model governed by $I(g, m)$ but they are also the Nahm equations [9] which arise in the problem of determining the moduli space of BPS monopoles, which are themselves solutions of the 3D Bogomolny equations.

They have been studied as the self-duality equations in YM classical mechanics [3] and it is in this context where the elliptic parameter, much more hidden than in the scalar case, arises. To unveil its physical meaning we shall consider the 'maximal' embedding of SU(2) in SU(3) given by the generators

$$\begin{aligned} E_+ &= \frac{1}{|\alpha^1|}(E_{\alpha^1} + E_{-\alpha^1}) + \frac{1}{|\alpha^2|}(E_{\alpha^2} + E_{-\alpha^2}) \\ E_- &= \frac{1}{|\alpha^2|}(E_{\alpha^1} - E_{-\alpha^1}) + \frac{1}{|\alpha^2|}(E_{\alpha^2} - E_{-\alpha^2}) \\ E_3 &= \frac{1}{|\alpha^1|^2}\alpha_a^1 H^a + \frac{1}{|\alpha^2|^2}\alpha_a^2 H^a \end{aligned}$$

where $E_{\mp\alpha^i}$ are step operators for the simple roots $\alpha^1=(1/2,\sqrt{3}/2)$, $\alpha^2=(1/2,\sqrt{3}/2)$ of SU(3) and $H^a$ are the generators in the Cartan subalgebra. In terms of the Gell-Mann $\lambda$ matrices it can easily be shown that $E_3=\mathrm{i}\lambda_3$, $E_1=(1/\sqrt{2})(E_+ + E_-)=(\mathrm{i}/\sqrt{2})(\lambda_4+\lambda_6)$ and $E_2=(1/\sqrt{2})(E_+ - E_-)=(\mathrm{i}/\sqrt{2})(\lambda_5-\lambda_7)$ satisfies $[E_i, E_j]=\varepsilon_{ijk}E_k$, the SU(2) commutation relations, simply by knowing the SU(3) structure constants $f_{ijk}$: $[\lambda_i, \lambda_j]2\mathrm{i}f_{ijk}\lambda_k$.

The ansatz $A_i(\varphi)=E_i f_i(\varphi)$, non-summation in i, converts (2.14) into the Euler equations for the spinning top: $\mathrm{d}f_1/\mathrm{d}\varphi=-2f_2f_3$ cyclically. These are, in Saviddy's scheme, the Bogomolny equations of the fundamental subsystem of YM classical mechanics: considering $A_i^a$ as a $3\times3$ matrix, our ansatz implies that the off-diagonal components are zero. The problem is mathematically equivalent to the motion of a 'particle', with coordinates $f_i$, moving in three dimensions under the potential $U=-\frac{1}{2}(f_2^2f_3^2+f_1^2f_2^2+f_1^2f_3^2)$. This is a 'chaotic' dynamical system [4], there are bounded trajectories which are not periodic, but the bounded solutions of the Euler equations, which are completely integrable, are periodic. An elliptic parameter appears by identifying the functions $f_i$ with the components of the angular velocity up to factors, the inertia momentum components, $f_i=I_i\omega_i$. In our case, we have $I_2=I_3$, $I_1=0$ and the rotation angle $\vartheta=2\varphi$. Then $k^2=1/I_2$ is the inverse of the inertia momentum and the limit $k^2=1$ is the separatrix between bounded and unbounded motion of the spinning top.

## 3. Elliptic solutions and their hyperbolic limit

We shall now consider the solutions of the equations we have discussed in the previous section. In the scalar case, a solution to (2.4) which we will call elliptic kink is (see figure 1)

$$\phi_K(\varphi, k)=\mathrm{sn}(\varphi, k). \tag{3.1}$$

The solution is periodic with periodicity dictated by the elliptic parameter $k^2$ through the condition $\phi_K(\varphi, k)=\phi_K(\varphi+4K(k), k)$ where

$$K(k)=\int_0^{\pi/2}\frac{1}{(1+k^2\sin^2\varphi)^{1/2}}\,\mathrm{d}\varphi$$

is the complete elliptic integral of the first time. In terms of more physical quantities the periodicity can be read from the relation $mL/(1+k^2)=4K(k)$. Because $K(0)=\pi/2$

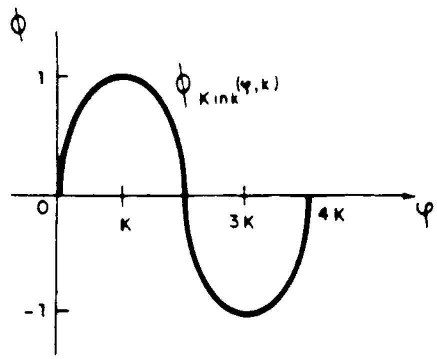


**Figure 1.** The elliptic kink.

and $K(1)=\infty$, $2\pi \leqslant mL \leqslant \infty$: there is no room for periodic solutions if the radius of the ring is too small!† The energy density is

$$\mathscr{E}(\lambda, m)=\frac{m^3}{\lambda}\frac{2k^2}{(1+k^2)^{3/2}}\mathrm{cn}^2\frac{m}{(1+k^2)^{1/2}}x_4\,\mathrm{dn}^2\frac{m}{(1+k^2)^{1/2}}x_4-\frac{m^3}{2\lambda}\frac{(1-k^2)^2}{(1+k^2)^{3/2}} \tag{3.2}$$

it is localised around the north pole ($\varphi=0$) and the south pole ($\varphi=2K$), with a width depending on $m$ (see figure 2), and there is a vacuum energy, the constant term, due to the renormalisation of $C$.

The solution (3.1) corresponds to the dynamical problem determined by (2.2); it is the trajectory depicted in figure 3 of a particle moving under the potential $-U$. The limit $k^2=1$ is the limit of infinite radius because then $K(1)=\infty$ or $L=\infty$. The solution, which ceases to be periodic,

$$\Phi_K(x_4)=\frac{m}{\lambda^{1/2}}\mathrm{sn}\left(\frac{m}{\sqrt{2}}x_{4,1}\right)=\frac{m}{\lambda^{1/2}}\tanh\left(\frac{m}{\sqrt{2}}x_4\right) \tag{3.3}$$

is a hyperbolic function; the separatrix between bounded and unbounded motion of the particle.

In any case there is a topological charge as a function of $k^2$

$$Q^{\mathrm{T}}(k^2)=\int \mathrm{d}\varphi\left(\frac{2k^2}{1+k^2}\mathrm{cn}^2\,\varphi\,\mathrm{dn}^2\,\varphi-\frac{1}{2}\frac{(1-k^2)^2}{1-k^2}\right) \tag{3.4}$$

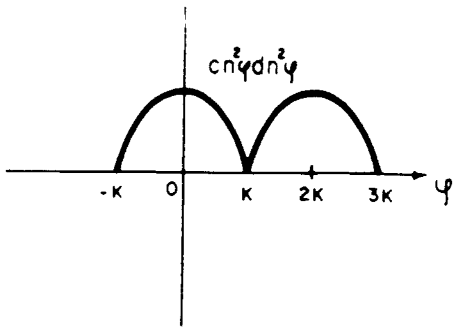


**Figure 2.** Energy density up to a factor and up to a constant.

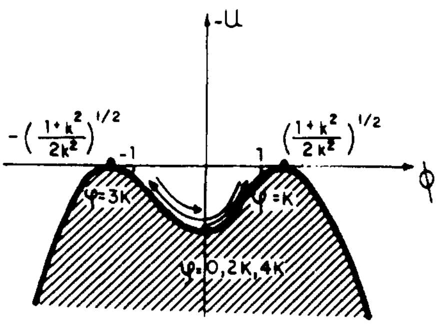


**Figure 3.** Periodic trajectory corresponding to the elliptic kink.

† This observation is due to A Actor.

which is in general infinite by integrating $\varphi$ between $-\infty$ and $\infty$. It is, however, finite if we restrict ourselves to a fundamental period $[0, 4K(k)]$ or, better, to a half-period $[-K, K]$. Observe that when the Bogomolny bound is saturated the energy is proportional to $Q^{\mathrm{T}}(k^2)$. In the limit $k^2=1$ we get the topological charge of the usual kink, $Q^{\mathrm{T}}(1)=1$. Therefore, as seen from the $(3+1+1)$-dimensional model, the solution will be a domain wall obtained by spatial translations of the localised energy density around the north pole in the interval $[-L, L]$. In the limit $k^2=1$ we get the Rubakov–Shaposhnikov domain wall.

The main idea of this paper is that Nahm equations (2.14) should be understood as 1D Bogomolny equations in the $(1+1)$-dimensional model with dynamics given by $I(g, m)$. Thus, their solutions are kinks, or solitons, in those models and domain walls in the original gauge systems. The solution proposed in [10] in terms of the Jacobi elliptic functions (see figure 4)

$$A_1^K(\varphi) = -E_1 k \operatorname{sn} \varphi \qquad A_2^K(\varphi) = \mathrm{i} E_2 k \operatorname{cn} \varphi \qquad A_3^K(\varphi) = \mathrm{i} E_3 \operatorname{dn} \varphi \tag{3.5}$$

closely resembles the elliptic kink of the scalar model. It is periodic, $A_i^K(\varphi) = A_i^K(\varphi + 4K(k))$, with periodicity in the $x_4$ variable given by $mL/(1+k^2)^{1/2} = 8K(k)$ for $4\pi \leqslant mL \leqslant \infty$. It describes a periodic trajectory of the fundamental subsystem of YM classical mechanics and represents a domain wall in the $(3+1+1)$-dimensional YM system of finite thickness, determined by the parameter $m$, of colour magnetic fields

$$\begin{aligned}
&F_{12}^K = -E_3 k^2 \operatorname{sn} \varphi \operatorname{cn} \varphi &\qquad &F_{\varphi 1}^K = -E_1 k \operatorname{cn} \varphi \operatorname{dn} \varphi \\
&F_{12}^K = E_2 k \operatorname{sn} \varphi \operatorname{dn} \varphi &\qquad &F_{\varphi 2}^K = -\mathrm{i} E_2 k \operatorname{sn} \varphi \operatorname{dn} \varphi \\
&F_{23}^K = \mathrm{i} E_1 k \operatorname{cn} \varphi \operatorname{dn} \varphi &\qquad &F_{\varphi 3}^K = \mathrm{i} E_3 k^2 \operatorname{sn} \varphi \operatorname{cn} \varphi .
\end{aligned} \tag{3.6}$$

$E(g, m)$ can be easily computed for our solutions. By saturating the Bogomolny bound the only contribution to (2.13) is the second integral, which must be 'topological' in origin; in fact, it is, because

$$\operatorname{tr}\left(\frac{\mathrm{d}A_i}{\mathrm{d}\varphi} \varepsilon_{ijk}[A_j, A_k]\right) = \mathrm{i} \operatorname{tr}(F_{\varphi i} \varepsilon_{ijk} F_{jk}) = 2k^2(1 - k^2 \operatorname{sn}^4 \varphi) \tag{3.7}$$

so that the integrand in $E(g, m)$ is proportional to the second Chern density and

$$E(g, m) = 4Q^{\mathrm{T}}(k^2) = 8k^2 \int \mathrm{d}\varphi (1 - k^2 \operatorname{sn} \varphi). \tag{3.8}$$

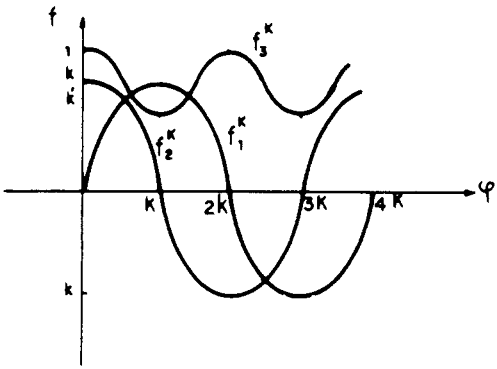


**Figure 4.** Elliptic YM kink.

As in the scalar case, translations of the energy density in spatial dimensions (although by (3.7) the density is concentrated around the north and south poles) produce the domain walls. The hyperbolic limit $k^2=1$ ($I_2=1$) is more involved. Then the solution is not periodic:

$$\begin{aligned} A_1^K(x_4) &= -\frac{m}{g}\,\mathrm{sn}\left(\frac{m}{\sqrt{2}}x_{4,1}\right)E_1 = -\frac{m}{g}\tanh\left(\frac{m}{\sqrt{2}}x_4\right)E_1 \\ A_2^K(x_4) &= \mathrm{i}\frac{m}{g}\,\mathrm{cn}\left(\frac{m}{\sqrt{2}}x_{4,1}\right)E_2 = \mathrm{i}\frac{m}{g}\,\mathrm{sech}\left(\frac{m}{\sqrt{2}}x_4\right)E_2 \\ A_3^K(x_4) &= \mathrm{i}\frac{m}{g}\,\mathrm{dn}\left(\frac{m}{\sqrt{2}}x_{4,1}\right)E_r = \mathrm{i}\frac{m}{g}\,\mathrm{sech}\left(\frac{m}{\sqrt{2}}x_4\right)E_3 \end{aligned} \tag{3.9}$$

and its topological charge

$$Q^{\mathrm{T}}(1) = \int \mathrm{d}x_4(1-\tanh^4 x_4) = \tfrac{8}{2} \tag{3.10}$$

which is fractional, tells us that it is a meron-like configuration†. As in the scalar kink case, the hyperbolic solution is a separatrix from unbounded motion, $k^2>1$. There is an important difference, however. In the scalar case the set of constant solutions of (2.4) is discrete. In the YM case constant matrices which commute with each other are also solutions of (2.14), a continuous set homeomorph to $S^1$ which plays an important rôle in connection with the stability of the kink-type solutions to be analysed in the next section‡.

## 4. Particle spectrum: the Lamé equation

To study the particle spectrum in the WKB approximation one must consider the Hessian operator arising from second variations of $S$ at critical points taken as classical backgrounds

$$\left.\frac{\delta^2 S}{\delta\Phi(x)\delta\Phi(y)}\right|_{\Phi_K} = \int\left(\partial_M\partial^M + \left.\frac{\delta^2 U}{\delta\Phi(x)\delta\Phi(y)}\right|_{\Phi_K}\right)\delta^5(x-y)\,\mathrm{d}^5x. \tag{4.1}$$

The linearised equations for small deformations are

$$\left(\partial_M\partial^M + \left.\frac{\delta^2 U}{\delta\phi(x)\delta\phi(y)}\right|_{\phi_K}\right)\chi(x)=0. \tag{4.2}$$

Because of $x_0$-invariance, the solution is a superposition of wavefunctions of the kind $\chi_\omega(x)=\mathrm{e}^{\mathrm{i}\omega x_0}\eta(\boldsymbol{x},\varphi)$ where $\eta$ is such that

$$h(\boldsymbol{x},\varphi)\eta(\boldsymbol{x},\varphi)=\omega^2\eta(\boldsymbol{x},\varphi) \qquad h(\boldsymbol{x},\varphi)=\left[-\left(\nabla^2+\frac{\partial^2}{\partial\varphi^2}\right)+6\phi_K^2(\varphi)-2\right]. \tag{4.3}$$

† It would be exactly a mean if the dependence of the magnetic fields on the other coordinates were of the $\delta$-function type.

‡ From the solution (3.5) one obtains new solutions by applying the symmetry transformations of the problem; essentially there exists invariance under two copies of SO(3), the first acting on the 'external' indices $A_i \to a_{ij}A_j$ with $a \in \mathrm{SO}(3)$, the second acting on the Lie algebra SO(3) generated by the $E_i$, the affine freedom $\varphi \to \alpha\varphi+\beta$ and, finally, the invariance under the Weyl group of the SU(3) root diagram.

Also, due to $\boldsymbol{x}$-invariance, $\eta(\boldsymbol{x}, \varphi) = \int (\mathrm{d}^3\boldsymbol{p}/2\omega)\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \vartheta(\varphi)$† is a solution of (4.3) if $\vartheta(\varphi)$ is an eigenfunction of the Sturm–Liouville operator $h(\varphi) = -\partial^2/\partial\varphi^2 + 6\phi_K^2(\varphi) - 2$, and the spectral problem one needs to solve is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} + 6k^2 \operatorname{sn}^2 \varphi - (1+k^2)\right) \vartheta_\lambda(\varphi) = \lambda^2 \vartheta_\lambda(\varphi) \tag{4.4}$$

which is the $n = 2$ generalised Lamé equation [5], the dispersion relation for the original small fluctuation $\chi$ being $\omega^2 = |\boldsymbol{p}|^2 + \lambda^2$.

We are dealing with quantum motion of a particle in the periodic potential $\mathcal{V}(\varphi) = 6k^2 \operatorname{sn}^2 \varphi - (1+k^2)$. The restriction of $\operatorname{sn} \varphi$ to real values is understood; this is required when one recognises the elliptic kink as a periodic trajectory of a particle. The differential operator in (4.4) is thus a Hermitian operator and the spectral condition for $\lambda$ [5]

$$\lambda^2 = (1+k^2) - k^2 \operatorname{sn}^2 \alpha_1 - k^2 \operatorname{sn}^2 \alpha_2 - 2 \operatorname{cn} \alpha_2 \operatorname{ds} \alpha_1 \operatorname{cn} \alpha_2 \operatorname{ds} \alpha_2 \tag{4.5}$$

where $\alpha_1$ and $\alpha_2$ are two complex spectral parameters, is admissible only for those values of the $\alpha_r$ which give real $\lambda$. The spectral parameters are linked by the relationship

$$\frac{\operatorname{sn} \alpha_2 \operatorname{cn} \alpha_1 \operatorname{dn} \alpha_1 + \operatorname{sn} \alpha_2 \operatorname{cn} \alpha_2 \operatorname{dn} \alpha_2}{\operatorname{sn}^2 \alpha_1 - \operatorname{sn}^2 \alpha_2} = 0 \tag{4.6}$$

determining the momentum of the travelling waves through the identity $\mathrm{i}p = -\Sigma_{r=1}^2 Z(\alpha_r)$, where $Z(\alpha_r)$ is the Jacobian zeta function. The eigenfunctions are

$$\vartheta_\lambda(\varphi) = \prod_{r=1}^{2} \frac{H(\varphi + \alpha_r)}{\Theta(\alpha_r)} \exp(-Z(\alpha_r)\varphi) \tag{4.7}$$

where $H$ and $\Theta$ are the first two Jacobi theta functions [5].

There are three allowed bands, real momentum, and two forbidden ones, imaginary momentum. The boundaries are critical values, with no propagation at all‡,

| $\lambda$ | $\vartheta_\lambda$ | |
|---|---|---|
| $\varepsilon_0 = 0$ | $\vartheta_0(\varphi) = \operatorname{dn} \varphi \operatorname{cn} \varphi$ | |
| $\varepsilon_1 = 1 + k^2 - 2\sqrt{1 - k^2 + k^4}$ | $\vartheta_1(\varphi) = \operatorname{cn}^2 \varphi - \dfrac{\alpha - \varepsilon_1 - (1+k^2)}{\varepsilon_1 + (1+k^2)}$ | |
| $\varepsilon_2 = 3k^2$ | $\vartheta_2(\varphi) = \operatorname{dn} \varphi \operatorname{sn} \varphi$ | (4.8) |
| $\varepsilon_3 = 3$ | $\vartheta_3(\varphi) = \operatorname{cn} \varphi \operatorname{sn} \varphi$ | |
| $\varepsilon_4 = 1 + k^2 + 2\sqrt{1 - k^2 + k^4}$ | $\vartheta_4(\varphi) = \operatorname{sn}^2 \varphi - \dfrac{2}{\varepsilon_4 + (1+k^2)}$ | |

the allowed and forbidden bands being $[\varepsilon_0, \varepsilon_1]$, $[\varepsilon_2, \varepsilon_3]$, $[\varepsilon_4, \infty)$ and $[\varepsilon_1, \varepsilon_2]$, $[\varepsilon_3, \varepsilon_4]$ (see figure 5).

We see that there are three types of fluctuations.

(i) The first is

$$\begin{aligned} &\chi(x^0, \boldsymbol{x}, \varphi) = \vartheta_{\varepsilon_\alpha}(\varphi) \exp(-\boldsymbol{p}\cdot\boldsymbol{x} + \mathrm{i}\omega x^0) \\ &\omega^2 = \boldsymbol{p}^2 + \varepsilon_\alpha \qquad \varepsilon_\alpha \in [\varepsilon_1, \varepsilon_2] \text{ or } [\varepsilon_3, \varepsilon_4] \text{ or } \varepsilon_\alpha = \varepsilon_0 \end{aligned} \tag{4.9}$$

the corresponding particles being confined inside the wall.

† This computation, and the general analysis of the spectrum of the Lamé equation, are due to J M Cerveró.
‡ Needless to say $\omega$ and $\boldsymbol{p}$ are dimensionless; to find $\underset{\sim}{\omega}$ and $\underset{\sim}{\boldsymbol{p}}$ one simply multiplies by $m$.

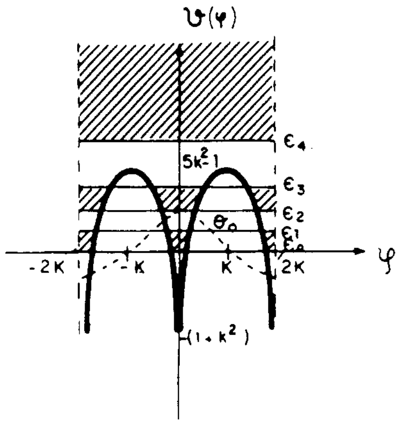


Figure 5. Band structure of the spectral problem (4.4). The allowed bands are shaded.

(ii) The second is

$$\chi(x^0, \boldsymbol{x}, \varphi) = \prod_{r=1}^{2} \frac{H(\varphi+\alpha_r)}{\Theta(\varphi)} \exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} + \mathrm{i}\rho\varphi + \mathrm{i}\omega x^0)$$
$$\omega^2 = \boldsymbol{p}^2 + \lambda^2 \qquad \lambda^2 \in (\varepsilon_0, \varepsilon_1) \text{ or } (\varepsilon_2, \varepsilon_3). \tag{4.10}$$

These particles are not confined; they move in the $x_4$ direction, tunnelling through the wall.

(iii) Similar to the previous case, the particles corresponding to $\lambda^2 > \varepsilon_4$ are completely free and escape observation, travelling in the fourth dimension.

Taking the Kaluza-Klein viewpoint, by which one understands the extra dimensions as describing internal properties of the particles, electric charge, for instance, fluctuations in those quantum numbers are not allowed in the first type. The other cases, particles with no fixed electric charge etc, would not be physical. In the hyperbolic limit $k^2 = 1$ the Rubakov-Shaposhnikov situation is recovered. The spectral condition reduces to

$$\lambda^2 = 4(\tanh\alpha_1 + \tanh\alpha_2)^2 = 4 + \rho^2. \tag{4.11}$$

Because $Z(\alpha_r)|_{k^2=1} = \tanh\alpha_r$, the 'continuous' eigenfunctions are

$$\vartheta_\lambda(\varphi) = \mathrm{e}^{\mathrm{i}\rho\varphi}(\varphi^2 - 3\mathrm{i}\rho\tanh\varphi - 1 + 3\tanh^2\varphi) \tag{4.12}$$

and the first two allowed bands collapse to the two bound states of the 'kink' potential [11].

In the YM case the dimensional reduction procedure described in section 2, on the usual 'Hessian' operator

$$h(x) = -\mathrm{D}_M \mathrm{D}^M + 2g^{MN}[F_{MN} \tag{4.13}$$

yields

$$h(\boldsymbol{x}, \varphi) = -\mathrm{D}_\varphi \mathrm{D}^\varphi + A_i A_i + 2[\partial_\varphi A_i - \mathrm{i}\varepsilon_{ijk} A_j A_k]\sigma_i \tag{4.14}$$

where the $\sigma_i$ are the Pauli matrices. At the periodic kink solutions, (4.14) takes the form

$$h(\varphi) = -\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} + 2k^2\,\mathrm{sn}^2\,\varphi - (E_3^2 + k^2 E_2^2) \tag{4.15}$$

a Sturm–Liouville operator $h = -(\mathrm{d}^2/\mathrm{d}\varphi^2)I + V(\varphi)$ where $V(\varphi)$ is the periodic potential

$$V(\varphi) = \begin{pmatrix} (2k^2\,\mathrm{sn}^2 - 1)k^2/2 & -k^2/2 & 0 \\ -k^2/2 & (2k^2\,\mathrm{sn}^2 - 1)k^2/2 & 0 \\ 0 & 0 & 2k^2\,\mathrm{sn}^2 - h^2 \end{pmatrix}. \tag{4.16}$$

After the above experience with the scalar model we know that the spectral problem $h\psi_\lambda(\varphi) = \lambda^2\psi_\lambda(\varphi)$ will be relevant to the particle spectrum of the YM system and hence we solve it by diagonalising $V(\varphi)$:

$$\begin{pmatrix} -\mathrm{d}^2/\mathrm{d}\varphi^2 + 2k^2\,\mathrm{sn} - k^2 & & 0 \\ 0 & -\mathrm{d}^2/\mathrm{d}\varphi^2 + 2k^2\,\mathrm{sn}^2 - 1 & 0 \\ 0 & & -\mathrm{d}^2/\mathrm{d}\varphi^2 - 2k^2\,\mathrm{sn}^2 - 1 - k^3 \end{pmatrix} \begin{pmatrix} \vartheta^1_\lambda(\varphi) \\ \vartheta^2_\lambda(\varphi) \\ \vartheta^3_\lambda(\varphi) \end{pmatrix}$$
$$= \lambda^2 \begin{pmatrix} \vartheta^1_\lambda(\varphi) \\ \vartheta^2_\lambda(\varphi) \\ \vartheta^3_\lambda(\varphi) \end{pmatrix} \tag{4.17}$$

the $n = 1$ Lamé equation appearing for the three critical values $k^2$, 1, $1 + k^2$. We shall classify the solutions of (4.17) into three types:

(a) $$\vartheta_0^{(1)}(\varphi) = \begin{pmatrix} \mathrm{dn}\,\varphi \\ 0 \\ 0 \end{pmatrix} \qquad \vartheta_{-1+k^2}(\varphi) = \begin{pmatrix} 0 \\ \mathrm{dn}\,\varphi \\ 0 \end{pmatrix} \qquad \vartheta_{-1}(\varphi) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{dn}\,\varphi \end{pmatrix}$$

(b) $$\vartheta_{1-k^2}(\varphi) = \begin{pmatrix} \mathrm{cn}\,\varphi \\ 0 \\ 0 \end{pmatrix} \qquad \vartheta_0^{(2)}(\varphi) = \begin{pmatrix} 0 \\ \mathrm{cn}\,\varphi \\ 0 \end{pmatrix} \qquad \vartheta_{-k^2}(\varphi) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{cn}\,\varphi \end{pmatrix}$$

$$\vartheta_1(\varphi) = \begin{pmatrix} \mathrm{sn}\,\varphi \\ 0 \\ 0 \end{pmatrix} \qquad \vartheta_{k^2}(\varphi) = \begin{pmatrix} 0 \\ \mathrm{sn}\,\varphi \\ 0 \end{pmatrix} \qquad \vartheta_0^{(3)}(\varphi) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{sn}\,\varphi \end{pmatrix}.$$

These are eigenfunctions for the special values $\lambda^2 = -1$, $-1 + k^2$, $-k^2$, 0, 0, 0, $k^2$, $1 - k^2$, 1 but restricted to $\lambda^2$ being real.

(c) There are also *bands* given by the spectral conditions

$$\lambda_1^2 = 1 - k^2\,\mathrm{sn}^2\,\alpha \qquad \lambda_2^2 = k^2 - k^2\,\mathrm{sn}^2\,\alpha \qquad \lambda_2^2 = -k^2\,\mathrm{sn}^2\,\alpha \tag{4.18}$$

for a complex parameter $\alpha$ which defines the momentum of the eigenfunctions

$$\vartheta_{\lambda_1}(\varphi) = \begin{pmatrix} \mathrm{e}^{-Z(\varphi)} \dfrac{H(\varphi+\alpha)}{\Theta(\varphi} \\ 0 \\ 0 \end{pmatrix} \qquad \vartheta_{\lambda_2}(\varphi) = \begin{pmatrix} 0 \\ \mathrm{e}^{-Z(\alpha)} \dfrac{H(\varphi+\alpha)}{\Theta(\varphi)} \\ 0 \end{pmatrix}$$

$$\vartheta_{\lambda_3}(\varphi) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^{-Z(\varphi)} \dfrac{H(\varphi+\alpha)}{\Theta(\varphi)} \end{pmatrix} \tag{4.19}$$

through the condition $Z(\alpha) = \mathrm{i}p$. In all three cases one has an allowed band, $Z(\alpha)$ purely imaginary, between the two first special eigenvalues; a forbidden one, $Z(\alpha)$

purely real, between the second and third special eigenvalues; and, finally, an allowed infinite band for values of $\lambda^2$ higher than the third. The situation, easier than the previous one in the scalar model in the sense that we are dealing with the $n = 1$ generalised Lamé equation, is depicted and summarised in figure 6.

In this case the Rubakov-Shaposhnikov analysis is more complicated. The whole Hessian presents three kinds of particle excitations. (a) The first has the form $A_i(\varphi, \boldsymbol{x}) = \int (\mathrm{d}^2p/2\omega)\vartheta_{\lambda_\alpha}(\varphi)e_i(p)\,\mathrm{e}^{\mathrm{i}p\cdot x}$ with $\vartheta_{\lambda_\alpha}$ in an allowed band of positive energy. These represent gluons with polarisation $e_i(\boldsymbol{r})$ which escape observation through the non-observable internal dimension, some of them by the tunnel effect, others by free motion. (b) Particles with $\vartheta_{\lambda_\alpha}(\varphi)$ in a forbidden band or in the boundary, $\rho = 0$, of the allowed bands, and the energy greater than zero. These are the real gluons we know to exist in the real world confined inside the wall according to the corresponding $\vartheta_{\lambda_i}$. Their dispersion relations in $\omega^2 = |\boldsymbol{r}|^2 + \lambda_\alpha^2$. (c) Particles with $\vartheta_{\lambda_\alpha}$ of negative $\lambda^2$. These are tachyons (see figure 7).

A clear understanding of the topological origin of the existence of tachyons, and hence of the unstability of periodic kinks, will provide a loophole for avoiding tachyonic pollution. Our claim is that tachyons appear as a consequence of the topology of the configuration space $\mathscr{C}$

$$\mathscr{C} = \{\text{set of } f_i : S^1 \to \mathbb{R} \mid f_i(\varphi) = f_i(\varphi + 4K) \text{ and } E(g, m) < +\infty\}.$$

It is a very well known fact that $\pi_k(\mathscr{C}) = \pi_k(\mathcal{M})$ [2], the order-$k$ homotopy group of $\mathscr{C}$ is equal to $\pi_k(\mathcal{M})$, where $\mathcal{M}$ is the vacuum orbit, the subset of $\mathscr{C}$ for which $E(g, m) = 0$

$$\mathcal{M} = \{f_1 = \mathrm{e}^{\mathrm{i}\alpha} \in S^1_{(K)} = \mathbb{R}/\mathbb{Z},\ f_2 = f_3 = 0\}.$$

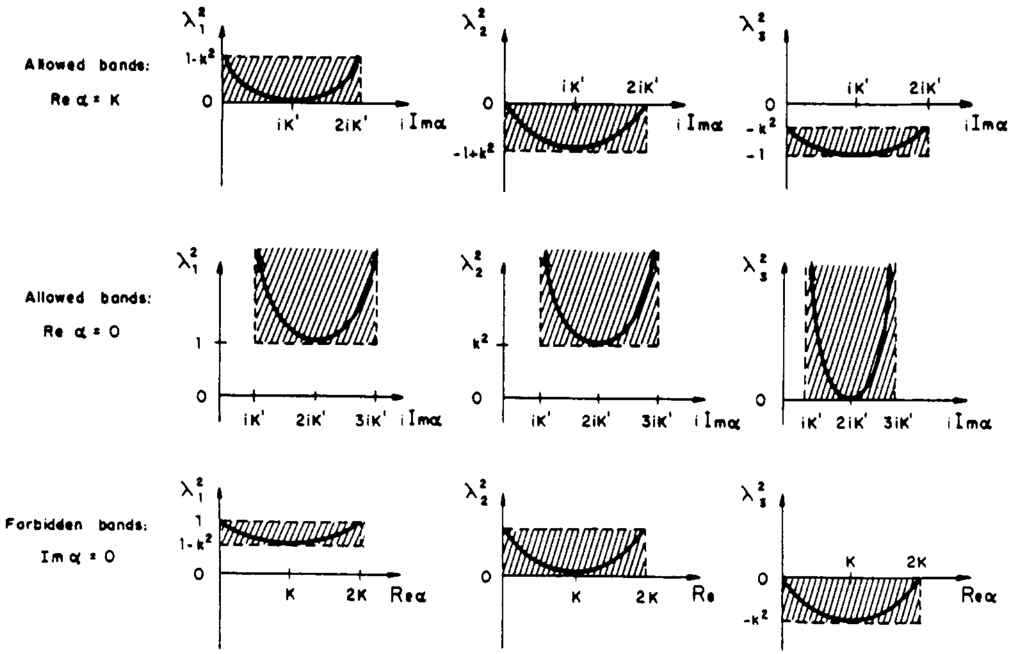


**Figure 6.** Energy bands in the YM case as a function of the complex parameter $\alpha$.

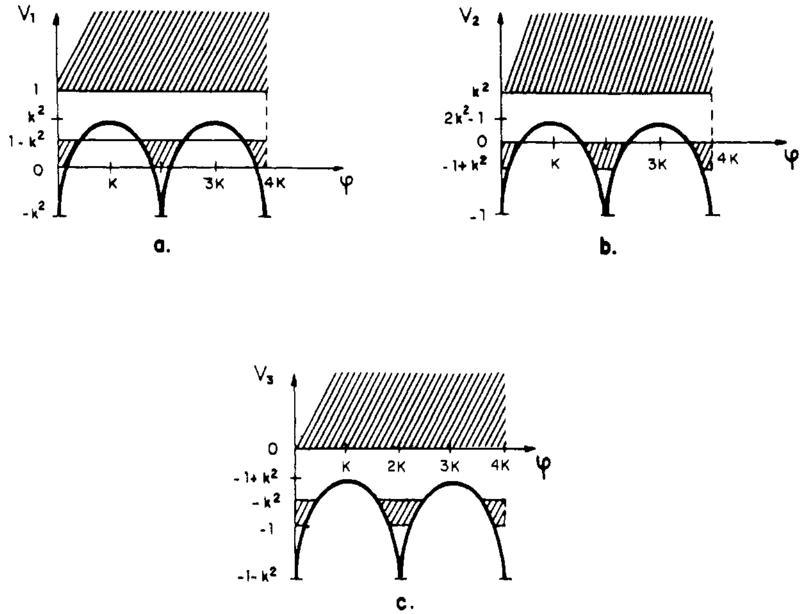


**Figure 7.** The potential wells in (4.17). The shaded areas correspond to allowed bands.

This is the set of constant solutions for which the potential energy of the fundamental subsystem of YMCM is zero. However, due to the fact that our functions $f_i$ take values really in $\mathbb{R}/\mathbb{Z}$, $\mathcal{M}$ is homeomorphic to $S^1$. Thus $\pi_1(\mathscr{C}) = \pi_1(\mathcal{M}) = \mathbb{Z}$ and the configuration space is non-simply connected.

The link between non-simply connectivity and saddle points—notice that because of (4.17) the periodic kinks are saddle points of $E(g, m)$, the tachyonic fluctuations obeying the directions of steepest descent—is provided by Ljusternik-Schnirelman theory [6]. We can construct a closed loop in $\mathscr{C}$

$$A_1(\varphi, \tau) = -E_1\left[\left(\frac{\cos \pi\tau - 1}{2}\right)(1 - f_\tau(\varphi) + 1\right]k$$

$$A_2(\varphi, \tau) = \mathrm{i}E_2 \sin \pi\tau(1 - f_\tau^2(\varphi))k \tag{4.20}$$

$$A_3(\varphi, \tau) = \mathrm{i}E_3 \sin \pi\tau(1 - k^2 f_\tau^2(\varphi))$$

which is non-contractible. It can be easily proved (see [12]) that for $\tau = 0 = 1$ $E(g, m, \tau)$ is a minimum while $E(g, m, \frac{1}{2})$ is a maximum as a function of $\tau$. For $\tau = 0 = 1$, the constant solution minimises $E(g, m)$ and the same can be proved for the periodic kink, $f_{1/2}(\varphi) = \mathrm{sn}\ \varphi$, at $\tau = \frac{1}{2}$. The loop (4.20) is therefore non-contractible and the reason why $A_1^K(\varphi)$ is a saddle point is the topology of $\mathscr{C}$.

The previous discussion may be interpreted in the following way: the dimensional reduction developed amounts to a spontaneous symmetry breakdown of the gauge symmetry; the gluons acquire a mass $\lambda_\alpha^2$, with a vacuum orbit $S^1$ and hence a Goldstone boson whose dynamics is governed by a kinetic term $\frac{1}{2}(\partial\alpha/\partial\varphi)^2$ hidden in $E(g, m)$. By adding a U(1) gauge field associated with the gauge transformation $\alpha'(\varphi) = \alpha(\varphi) + \beta(\varphi)$, $\chi'(\varphi) = \chi(\varphi) + \mathrm{i}\, \mathrm{d}\beta/\mathrm{d}\varphi$, and substituting the kinetic term by $\frac{1}{2}(\mathrm{d}\alpha/\mathrm{d}\varphi + \mathrm{i}\chi(\varphi))^2$, we fix the U(1) freedom by picking a 'unitary' gauge. There is thus no longer the freedom of choosing any point of $S^1$ as a vacuum state and the topology of the configuration space, modified by adding the new gauge field, is trivial, the tachyons disappearing.

We end this section by showing a non-obvious link between the periodic kink of the YM system and the elliptic kink of the $\lambda\phi^4$ model. The ansatz

$$f_1(\varphi) = kf(\varphi) \qquad f_2(\varphi) = \frac{k}{\sqrt{2}}(1 - f^2(\varphi))^{1/2} \qquad f_3(\varphi) = \frac{k}{\sqrt{2}}(1 - k^2 f^2(\varphi))^{1/2} \tag{4.21}$$

makes the spinning top equations tantamount to

$$\frac{\mathrm{d}f}{\mathrm{d}\varphi} = (1 - f^2(\varphi))^{1/2}(1 - k^2 f^2(\varphi))^{1/2} \tag{4.22}$$

the Bogomolny equations for the elliptic kink. This is remarkable for two reasons: (i) the limit between periodic and unbounded behaviour of the asymmetric top is traced back to the same limit in a very simple mechanical system; (ii) kinks are solutions of a dimensional reduction of the self-duality equations of the four-dimensional Yang–Mills system, in a similar sense to BPS magnetic monopoles and vortices [13]. Moreover, the ansatz (4.21) means that the fundamental subsystem of YMCM contains as integrable subsystem one which is equivalent to a generalisation of the MSTB model, three scalar fields with interactions given by [12]

$$U - \tfrac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 - 1)^2 - \tfrac{1}{4}(\phi_2^2 + \phi_3^2)$$

the unstable kink of this model corresponding to the periodic kink in YM.

## 5. Chromo-electric and chromo-magnetic membranes and strings

The periodic kink solution described in the previous section can be interpreted as having another physical meaning. A different choice of the character of the coordinates will do the job. Consider now the YM action defined on a 4-torus $T^5$

$$S = \frac{1}{2g^2}\int \mathrm{d}\varphi\, \mathrm{d}\varphi_1\, \mathrm{d}\varphi_2\, \mathrm{d}\varphi_3\, \mathrm{tr}(F_{\varphi\varphi_i}F_{\varphi\varphi_i} + F_{\varphi_i\varphi_j}F_{\varphi_i\varphi_j}) \tag{5.1}$$

where $\varphi$ is a (angular) spatial coordinate and where we choose one of the $\varphi_i$, $i = 1, 2, 3$, let us say $\varphi_3$, as 'Euclidean' time. The Pontryagin number is

$$k = \frac{1}{4\pi^2}\int \mathrm{d}\varphi\, \mathrm{d}\varphi_1\, \mathrm{d}\varphi_2\, \mathrm{d}\varphi_3\, \mathrm{tr}(F_{\varphi_1\varphi_3}F_{\varphi_3\varphi} + F_{\varphi_1\varphi_3}F_{\varphi_2\varphi} + F_{\varphi_2\varphi_3}F_{\varphi_1\varphi}). \tag{5.2}$$

In the axial gauge $A_\varphi = 0$ and considering only gauge potentials which depend on $\varphi$, the self-duality equations reduce to

$$F_{\varphi\varphi_i} = \varepsilon_{ijk}F_{\varphi_j\varphi_k} \tag{5.3}$$

where $F_{\varphi\varphi_i} = \mathrm{d}A_{\varphi_i}/\mathrm{d}\varphi$ and $F_{\varphi_i\varphi_j} = -\mathrm{i}[A_{\varphi_i}, A_{\varphi_j}]$. These equations are mathematically identical to (2.14) and thus they admit

$$A^k_{\varphi_1}(\varphi) = -E_1 k \,\mathrm{sn}\,\varphi \qquad A^k_{\varphi_2}(\varphi) = \mathrm{i}E_2 k \,\mathrm{cn}\,\varphi \qquad A^k_{\varphi_3}(\varphi) = \mathrm{i}E_3 \,\mathrm{dn}\,\varphi \tag{5.4}$$

as solutions of the periodic kink type. These kinks carry both colour-magnetic fields, $F_{\varphi_\alpha\varphi}$, $F_{\varphi_\alpha\varphi_\beta}$, $\alpha, \beta = 1, 2$, and colour-electric fields, $F_{\varphi\varphi_3}$, $F_{\varphi_\alpha\varphi_3}$, if $\varphi_3$ is analytically continued to normal time, as given by (3.6). Keeping the independence in the $\varphi_\alpha$ coordinates and translating the solutions in these directions produces a 'membrane' of constant energy density which can be made as thin as we wish in the $\varphi$ coordinate by varying the parameter $m$. Notice that we can reshuffle the coordinates $\varphi$, $\varphi_1$, $\varphi_2$ to obtain membranes in planes orthogonal to any of them.

Due to the stability analysis previously performed, which is also valid with this new interpretation of the solution, we know that, besides the elementary excitations, the quantum version of the reduced YM model possesses one eigenstate of the Hamiltonian corresponding to the quantum kink [14]. One may therefore obtain quantum membranes in the YM spectrum which have a quantum behaviour of their own according to Polyakov's philosophy [15].

There is still another interesting possibility of dimensional reduction (different choice of the character of the $\varphi_i$). Let us consider everything in the previous situation to be independent of $\varphi_3$ and let us choose $A_{\varphi_3} = \phi$ where $\phi$ is a map from $T^3$ in $S^2$, i.e. a Higgs field in the Prasad–Sommerfield limit. The action (5.1) reduces to

$$S = \frac{m}{2g^2} \int \mathrm{d}\varphi\, \mathrm{d}\varphi_1\, \mathrm{d}\varphi_2\, \mathrm{d}\varphi_3\, \mathrm{tr}(F_{\varphi\varphi_\alpha} F_{\varphi\varphi_\alpha} + F_{\varphi_\alpha\varphi_\beta} F_{\varphi_\alpha\varphi_\beta} + \mathrm{D}_\varphi\phi \mathrm{D}_\varphi\phi + \mathrm{D}_{\varphi_\alpha}\phi \mathrm{D}_{\varphi_\alpha}\phi) \tag{5.5}$$

and the magnetic charge, coming from the Pontryagin number, see [14], is

$$\mu = \frac{1}{2} \int \mathrm{d}\varphi\, \mathrm{d}\varphi_1\, \mathrm{d}\varphi_2\, \mathrm{tr}(F_{\varphi_1\varphi_2} \mathrm{D}_\varphi\phi + F_{\varphi_2\varphi} \mathrm{D}_{\varphi_1}\phi + F_{\varphi_1\varphi} \mathrm{D}_{\varphi_2}\phi). \tag{5.6}$$

The self-duality equations, now the Bogomolny equations for monopoles,

$$\mathrm{D}_\varphi\phi = \varepsilon_{\alpha\beta} F_{\varphi_\alpha\varphi_\beta} \qquad \mathrm{D}_{\varphi_\alpha}\phi = \varepsilon_{\alpha\beta} F_{\varphi_\beta\varphi} \tag{5.7}$$

are again equivalent to equation (2.14), in the gauge $A_\varphi = 0$ and when both $A_{\varphi_\alpha}$ and $\phi$ are only functions of $\varphi$. The periodic kink, now translated into $\varphi_1$ or $\varphi_2$, depending on which coordinate we take as Euclidean time, is a string carrying both colour-electric and colour-magnetic fields (see figure 8). The electric and magnetic configuration

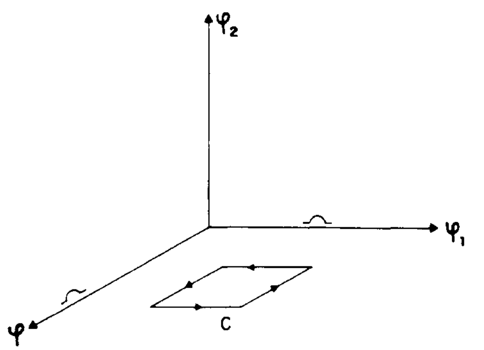


**Figure 8.** A Wilson loop formed by two pieces of strings; both of them are colour-electric and colour-magnetic tubes.

shown in figure 8 is created by the Wilson and 't Hooft operators

$$
\begin{aligned}
A(C) &= \operatorname{Tr} P \exp\left(\mathrm{i}\int_C (A_\varphi\, \mathrm{d}\varphi + A_{\varphi_1}\, \mathrm{d}\varphi_1)\right) \qquad & E_\varphi &= \frac{\mathrm{d}A_{\varphi_2}}{\mathrm{d}\varphi} \\
B(C) &= \operatorname{Tr} P \exp\left(\mathrm{i}\int_C (E_\varphi\, \mathrm{d}\varphi + E_{\varphi_1}\, \mathrm{d}\varphi_1)\right) \qquad & E_{\varphi_1} &= -\mathrm{i}[A_{\varphi_1}, A_{\varphi_2}].
\end{aligned}
\tag{5.8}
$$

Because the colour-electric and colour-magnetic fluxes in $C$ form tubes analogous to those existing in Type-II superconductivity, due to the existence of the periodic kinks, both $A(C)$ and $B(C)$ satisfy the area law criterion of confinement. In other words [16], the system presents both colour-electric and colour-magnetic order.

## 6. Further comments

We shall devote this last section to discussing how the model can account for massless fermions in a natural way and to explaining how this kind of structure may exist in other physical models.

The key idea for the existence of massless fermions in the model is taken from a similar situation in the (1+1)D Jackiw–Rebbi model [17]; in the presence of a kink there are fermionic zero modes. In our model the Dirac equation

$$
\mathrm{i}\Gamma_M \mathrm{D}^M \psi = 0 \qquad \Gamma_5 = \gamma_5 \tag{6.1}
$$

for chiral fields $\psi_{\mathrm{L}} = \frac{1}{2}(1+\gamma_5)\psi$ depending only on the $\varphi$ coordinate, in the presence of periodic kinks reduces to

$$
\mathrm{i}\partial_\varphi \psi_{\mathrm{L}}^\alpha(\varphi) = -\mathrm{i}g\gamma_i A^K_{i\alpha\beta}(\varphi)\psi_{\mathrm{L}}^\beta(\varphi) \tag{6.2}
$$

where $\alpha, \beta = 1, 2, 3$ are 'colour' indices. It can be easily proved that

$$
\mathring{\psi}_{\mathrm{L}}^1(\varphi) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}g\,\mathrm{Am}\,\gamma} \\ \mathrm{e}^{\mathrm{i}g\,\mathrm{Am}\,\varphi} \end{pmatrix} \qquad \mathring{\psi}_{\mathrm{L}}^2(\varphi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}g\,\mathrm{Am}\,\varphi} \\ \mathrm{e}^{-\mathrm{i}g\,\mathrm{Am}\,\varphi} \end{pmatrix} \qquad \mathring{\psi}_{\mathrm{L}}^3(\varphi) = 0 \tag{6.3}
$$

where $\mathrm{Am}\,\varphi = \mathrm{cn}\,\varphi + \mathrm{i}\,\mathrm{sn}\,\varphi$, are solutions of (6.2). Thus, a fermionic zero mode, the colour triplet

$$
\mathring{\psi}_{\mathrm{L}}(\varphi) = \begin{pmatrix} \mathring{\psi}_{\mathrm{L}}^1(\varphi) \\ \mathring{\psi}_{\mathrm{L}}^2(\varphi) \\ \mathring{\psi}_{\mathrm{L}}^3(\varphi) \end{pmatrix} \tag{6.4}
$$

exists in the presence of periodic kinks. The massless fermion fields in the cylinder will have the form

$$
\psi_{\mathrm{L}}(\varphi, x) = \mathring{\psi}_{\mathrm{L}}(\varphi) \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{m}{E}\left(a_{-1/2}(\boldsymbol{k})u_{1/2}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot x - \mathrm{i}Et} + b^+_{1/2}(\boldsymbol{k})v_{1/2}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot x + \mathrm{i}Et}\right) \tag{6.5}
$$

and are thus confined inside the wall while freely propagating in Minkowski space.

If we look at the solutions (6.3) from the point of view of the models where the periodic kinks have real physical meaning as strings or membranes, we find massless fermions at their cores. As in the Rubakov effect [18], a strong violation of the fermionic number can be produced by our strings or membranes, a mechanism reminiscent of the Witten proposal for superconducting strings [7].

As a final comment there is hope of obtaining these fascinating structures in other kinds of physical models. For example, gauge theories with a gauge group SU(2), like the Weinberg-Salam model, have the same kind of periodic kinks with a particle spectrum given by the $n=\frac{1}{2}$ Lamé equation. In the opposite sense, SU(5) GUT would permit periodic kinks with a particle spectrum based on the much more difficult case of the $n=2$ Lamé equation.

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